Hardness of computation of quantum invariants on 3 manifolds with restricted topology

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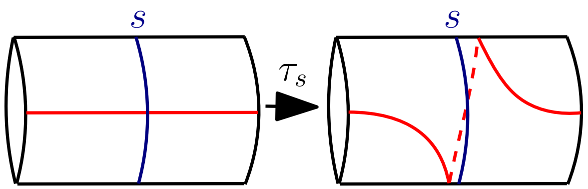
Complexity of invariants

Quantum invariants are strongly distinguishing invariants, usually defined for knots and **3-manifolds**. Approximating them is generally hard, so it is natural to look at whether some **restrictions to the topology** make the problem easier.

Samperton (2023), for example, showed that approximations of some knot invariants stay hard even when restricted to **hyperbolic** or minimally presented knots. We show that this is also the case for 3-manifolds.

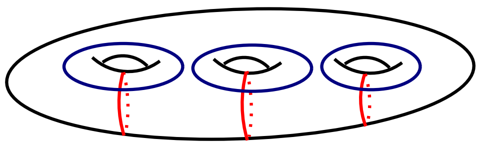
The curve graph

The **curve graph**, $C(\Sigma_g)$, of a closed surface of genus g has each vertex representing an isotopy class of **essential curves**, with vertex connected whenever the elements can be made disjoint. Homeomorphisms on the surface, such as **Dehn twists**, act on the curve graph.



Heegaard diagrams

Every closed 3-manifold M can be expressed by a **Heegaard diagram**, that is, two sets of g disjoint curves, α and β, in a closed surface, Σ_g, which describe how to glue two compact manifolds of boundary Σ_g to form M.



The **Hempel distance** of a Heegaard splitting, $d(\alpha, \beta)$, is the edge distance of α and β in $C(\Sigma_g)$. If $d(\alpha, \beta) \geq k \geq 3$, then

1. (α, β) is strongly irreducible;
2. M is hyperbolic;
3. an orientable and incompressible surface of genus at most $2k$ cannot be embedded in M.

Vafa's theorem

For every **Reshetikhin–Turaev (RT) invariant**, there exists a constant $N \in \mathbb{Z}^+$ such that, by applying any Dehn twist a multiple of N-times to a Heegaard diagram, the invariant of the represented manifold is conserved.

Our contribution

Theorem (HE and CM, 2025): Let (α, β) be a Heegaard diagram represeting a manifold M. For a fixed k and RT invariant, one can find in **polynomial time** a new Heegaard diagram (α', β') with $d(\alpha', \beta') \geq k$ and representing a manifold of same RT invariant of M.

Idea of the proof: provided $d(\alpha, \beta)$ is small, if s is a curve in Σ_g such that $d(s, \alpha) + d(s, \beta) > k + 2$, then $d(\alpha, \tau_s^m(\beta)) \geq k$ for $m \geq k$ (Yoshizawa, 2014). We make $m = Nk$ and describe an algorithm to construct the curve s.

Alagic and Lo (2017) found a family of RT invariants that approximating within good accuracy is $\#P$-hard. Our result implies that the problem is still $\#P$-hard for manifolds with properties (1)-(3). Similar construction implies $\#P$-hardness of approximating the associated **Turaev-Viro invariant**.

References

1. Eric Samperton. Topological quantum computation is hyperbolic. *Communications in Mathematical Physics*, 402:79–96, 2023.
2. Michael Yoshizawa. High distance Heegaard splittings via Dehn twists. *Algebraic & GeometricTopology*, 14(2):979–1004, 2014.